Discrete Mathematics 72 (1988) 89–92
North-Holland

A NOTE ON DISTANCE-DOMINATING CYCLES

P. FRAISSE
LRI, UA 410 du CNRS, Université Paris Sud, Bât 490, 91405 Orsay Cedex, France

Recevied 12 May 1987

Nous prouvons une conjecture due à Bondy et Fan. Un cycle C d'un graphe G est dit m-dominant si tout sommet de $V(G-C)$ est à distance au plus m de C. Notre résultat est: si G est k-connexe, et si G n'a pas de cycle m-dominant, alors il existe un stable de cardinal $k+1$, dont les sommets sont deux à distance $em+2$ au moins.

We prove a conjecture of Bondy and Fan. Let C denote a cycle of a graph G. We say that C is m-dominating if all the vertices of $V(G-C)$ are at a distance at most m from C. Our result is: if G is k-connected and has no m-dominating cycle, then there is an independent set of cardinality $k+1$, whose vertices are pairwise at a distance at least $2m+2$.

We consider throughout the paper finite, non-oriented graphs without loops. Let H be a subgraph of a given graph G. Denote by $V(H)$ and $E(H)$ the vertex and edge set of H. An H-path is a path of G whose endvertices are in $V(H)$ and whose inner vertices are not in $V(H)$. If a and b are two vertices of H an a–b path in H is a path whose endvertices are a and b, and whose inner vertices are in $V(H)$. The length of a path P is the number of edges of this path. We abbreviate $|P| = |V(P)| = E|(P)| + 1$.

For $K, L \subset V(G)$, $N_K(L)$ denotes the set of vertices of K adjacent to some vertex in L. $N_K(\{x\})$ will be denoted shortly by $N_K(x)$, and $N_G(L)$ by $N(L)$. The cardinality of $N_K(x)$ is denoted by $d_K(x)$. We use the same notations if K or L are subgraphs of G.

Let us consider a cycle $C = (c_1, c_2, \ldots, c_p, c_1)$. A direction of traversing of C will be a choice between increasing or decreasing order on the indices. Suppose we choose the increasing order. Then $C[c_i, c_j]$ denotes the subpath $(c_i, c_{i+1}, \ldots, c_{j-1}, c_j)$, the indices being taken modulo p. $C[c_i, c_j)$, $C(c_i, c_j]$ and $C(c_i, c_j)$ are obtained from $C[c_i, c_j]$ by deleting respectively c_j, c_i and $\{c_i, c_j\}$. We set $c_i^+ = c_{i+1}$.

Let $d(x, y)$ denote the length of a shortest path from x to y in G, and if K denote a subgraph of G, $d(x, K) = \min\{d(x, y)/y \in V(K)\}$. A cycle C is m-dominating if and only if for all $x \in V(G-C)$, $d(x, C) \leqslant m$. Set $R_m(v) = \{u \in V(G) - v/d(u, v) \leqslant m\}$. In [1] Bondy and Fan suggested the following conjecture:

Conjecture 1. If G is k-connected, then either G has an m-dominating cycle or an independent set $\{x_0, \ldots, x_k\}$ such that for $i \neq j$,

$$R_m(x_i) \cap R_m(x_j) = \varnothing \quad \text{and} \quad \sum_{i=0}^{k} |R_m(x_i)| < n - 2k.$$

We prove in fact a little more:

Lemma 2. *Assume G is k-connected. If an independent set $\{x_0, \ldots, x_k\}$ is such that for $i \neq j$, $R_m(x_i) \cap R_{m+1}(x_j) = \varnothing$, then $\sum_{i=0}^{k} |R_m(x_i)| < n - 2k$.*

Proof. As G is k-connected, each cutset of G has at least k vertices. Let K denote the set of vertices at distance exactly $m+1$ from x_0. Then K is a cutset: each path from x_0 to x_1 must contain a vertex of K, as $d(x_0, x_1) > 2m+1$. The sets $R_m(x_i)$, $0 \leqslant i \leqslant k$ being disjoint, and containing no vertex of K, and no vertex x_i, we find $n \geqslant \sum_{i=0}^{k} |R_m(x_i)| + 2k + 1$. □

Theorem 3. *If G is k-connected, then either G has an m-dominating cycle or an independent set $\{x_0, \ldots, x_k\}$ such that for $i \neq j$, $R_m(x_i) \cap R_{m+1}(x_j) = \varnothing$.*

Proof. Assume G has no m-dominating cycle. If C is a cycle of G then set $F(C) = \{x \in V(G - C) / d(x, C) > m\}$. Let C denote a cycle of G such that

(1) $|F(C)|$ is as small as possible.

With this assumption, there is a component H of $G - C$ with $F(C) \cap H \neq \varnothing$. We assume:

(2) C is chosen so that $|H|$ is as small as possible.

Let $x_0 \in F(C) \cap H$. Fix a maximum set of disjoint paths Q_q, $1 \leqslant q \leqslant p$ from x_0 to C, and let A denote the endvertices of these paths on C. By Condition 2, A does not contain two consecutive vertices of C. Then $|A| \geqslant k$ by Menger's theorem. Fix an orientation on C, and assume that $A = \{a_1, \ldots, a_p\}$, with $a_1 \cdots a_p$ in this order on C. For each i, j $i \neq j$, there is an $a_i - a_j$ C-path $Q_{i,j} = Q_i Q_j$, internally in H, because H is connected.

Consider two indices i, j not necessarily distinct. Let $K_{i,j}$ denote the set of cycles C' of G such that:

(3) $V(C') \cap V(H) \neq \varnothing$.

(4) $V(C) - V(C') = C(a_i, v_i(C') \cup C(a_j, v_j(C'))$, with $v_q(C') \in C(a_q, a_{q+1}]$, $q = i, j$.

Remark that if C' satisfies Conditions 3, then $F(C') - F(C) \neq \varnothing$, else C' would contradict Condition 2. $K_{i,j}$ is not empty, as the cycle $C_i = C[a_{i+1}, a_i]Q_{i,i+1}$ belongs to $K_{i,j}$. For each cycle C' in $K_{i,j}$, $F(C') - F(C) \neq \varnothing$ implies that, as $|F(C)|$ is minimum, there exist a vertex x_q such that $d(x_q, C) \leqslant m$, and $d(x_q, C') > m$. By Condition 4, it means that there is a path (of length at most m) from x_q to $C(a_i, v_i(C'))$ or to $C(a_j, v_j(C'))$ with no inner vertex in $V(C) \cup V(C')$. If the first possibility holds, and if moreover no cycle C'' of $K_{i,j}$ is such that

$V(C'') \cap V(C)$ strictly includes $V(C') \cap V(C)$, then we say that C' belongs to a set $L_{i,j}$: so we summarize that by the two following conditions:

(5) There is a path from $x_q \in F(C') - F(C)$ to $C(a_i, v_i(C'))$, with no inner vertex in $V(C) \cup V(C')$.

(6) $C(a_i, v_i(C')) \cup C(a_j, v_j(C'))$ is minimal for inclusion.

Mark that if C' is in $L_{i,j}$ and $x_q \in H$, then we can construct a new cycle C'' contradicting Condition 6: we use the path from x_q to $v \in C(a_i, v_i(C')$ of Condition 5 and the fact that a_i is connected to a vertex of H, to form a cycle C'' intersecting C on $C[v, a_i]$. Thus $x_q \notin H$.

The existence of C_i proves $L_{i,i} \neq \emptyset$: indeed for each cycle of $K_{i,i}$, there must be a minimal cycle in $L_{i,i}$, as Condition 5 is a consequence of Condition 4 when $i = j$.

Assume $C' \in L_{i,j}$. Let $u(x_q)$ denote a vertex of $C(a_i, v_i(C'))$ joined to x_q by a path with no inner vertex in $V(C) \cup V(C')$, such that $|C(a_i\, u(x_q)]|$ is minimum. Choose C' in $\bigcup_{j=1}^{k} L_{i,j}$ such that:

(7) $|C(a_i\, v_i(C'))|$ is minimum,

and let x_i be defined as x_q above, and set $u(x_i) = u_i$.

Now we have constructed the set of $(k+1)$ independent vertices of the theorem, as we prove below.

Proposition 4. $d(x_0, x_i) > 2m + 1, \quad 1 \leq i \leq k.$

Proof. Indeed each vertex of C is at a distance at least $m+1$ from x_0. Let C' be the cycle of $L_{i,j}$ defining x_i. Assume there is a path of length less than $2m+2$ between x_0 and x_i. As $x_i \notin H$ this path P must contain a vertex of C. This vertex, denoted by u, must be on $V(C) - V(C')$, otherwise P would be longer than $2m+1$. Say u is on $C(a_q, v_q(C'))$, $q = i$ or j, and let Q denote the path between a_q and u obtained by following Q_q and P. Then the cycle $C'' = C[u, a_q]Q$ contradict Condition 6 for C'. □.

Proposition 5. $\forall i, j \quad 1 \leq i < j \leq k, \qquad d(x_i, x_j) > 2m + 1.$

Proof. Assume that x_i and x_j are defined by the cycles C_i' and C_j'. Assume there is a path P containing no vertex of C between x_i and x_j, or that $x_i = x_j$. Then there must be a path Q with no inner vertex of C between u_i and u_j. This path may not contain vertices of H, as x_i does not belong to H. Thus the cycle $C'' = C[u_j, a_i]Q_{i,j}C[u_i, a_j]Q$ contradicts the definition of x_i or x_j: C'' is in $K_{i,j}$, then Condition 5 is verified for i [resp for j], contradicting Condition 7.

So suppose that there is a path P of length at most $2m+1$ between x_i and x_j, and let y_i and y_j denote the first vertex of C on this path, starting from respectively x_i and x_j. We may assume that $|P(x_i\, y_i]| < m$. Then $y_i \in C(u_i, v_i(C_i')) \cup C(a_{j+1}, v_j(C_i'))$.

Case 1. $y_j \in C(u_i, v_i(C_i'))$. Then there is a path Q between u_j and y_j, with no inner vertex in H nor in C. Moreover and the cycle $C'' = C[u_j, a_i]Q_{i,j}C[y_j, a_j]Q$ cannot be in $L_{i,j}$ by Condition 7. It cannot be in $L_{j,i}$ by Condition 7. A cycle containing $V(C) \cap V(C'')$, verifying Condition 6, contradicts also Condition 7, and thus the choice of x_i and x_j. As, by the existence of C'', there must be such a cycle, we reach a contradiction.

Case 2. $y_j \in C(a_{j+1}, v_j(C_i'))$. As $d(x_i, y_j) \leqslant m$, we deduce that there must be no vertex of $V(C_i')$ on $P(x_i, y_j)$, and so there is a path between y_j and $C(a_i, v_i(C_i'))$. But such a path contradicts Condition 6. □

This ends the proof of the theorem, since $d(x_i, x_j) \geqslant 2m + 2$ is equivalent to $R_m(x_i) \cap R_m(x_j) = \emptyset$. Together with Lemma 2, it proves Conjecture 1. It is sharp, as it is proved by the following example: take a star $K_{1,k+1}$, replace each edge by a path of length $m + 1$, and replace all the vertices but the leaves by a complete graph K_k. This graph is k-connected, and the leaves form an independent set of order $k + 1$, each pair of leaves being at distance $2m + 2$. It has no m-dominating cycle, and is maximal so: adding a new edge creates an m-dominating cycle.

References

[1] J.A. Bondy and G.H. Fan, A sufficient condition for dominating cycles, preprint.